## Recycling of quantum information: Multiple observations of quantum clocks

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(7 May 2000)

How much information about the original state preparation can be extracted from a quantum system which already has been measured? That is, how many independent (non-communicating) observers can measure the quantum system sequentially and give a nontrivial estimation of the original unknown state? We investigate these questions and we show from a simple example that quantum information is not entirely lost as a result of the measurement-induced collapse of the quantum state, and that an infinite number of independent observers who have no *prior* knowledge about the initial state can gain a partial information about the original preparation of the quantum system.

PACS numbers: 03.65.Bz, 42.50.Ar

From the deterministic measurement model employed in classical physics it follows that the state of the physical system is not affected by measurement. That is, information about states of the system can be determined with an arbitrary precision. Formally, from a kinematical point of view, this can be expressed as follows: in classical physics there are measurements (m) for which the statistics of the measurement results (r) characterized by the conditional probability distribution  $p_m(r|s)$  can be, for all possible states s of the given classical system, of the form

$$p_m(r|s) = \delta(s_r - s). \tag{1}$$

Moreover, these measurements do not change the state of the classical system, so an arbitrary number of independent observers (i.e. observers who do not communicate) can determine the state.

The standard Copenhagen interpretation of quantum mechanics is deeply rooted in a model of non-deterministic statistical measurement [1]. From the kinematical point of view the quantum theory models (predicts) the statistics of results registered by a measuring device when the measurement is performed on a quantum object. Within this non-deterministic model of measurement the conditional probability distribution  $p_m(r|s)$  can never be of the form (1) for arbitrary initially unknown states of a quantum system. In quantum mechanics the conditional probability distribution  $p_m(r|s)$  is given by the expression

$$p_m(r|s) = \text{Tr}\left[\hat{O}_r\hat{\rho}_s\right],$$
 (2)

where the set of positive operators  $\hat{O}_r$  which sum up to the identity operator (i.e., the POVM) models the measuring device and the density matrix characterizes the state of the quantum object being subject of the measurement.

The axiomatics of quantum theory implicitly require that the state of the system is changed during measurement. Otherwise, repeated measurements of the previously measured but unchanged quantum state could reveal yet more information about the state. Consequently, the measurement model would eventually be equivalent to the standard deterministic measurement model of classical physics. Therefore there is an additional rule which excludes the possibility of repeated measurements. This additional principle is the well known von Neumann projection postulate.

Nevertheless it is an interesting question to ask how much information about the original state is "left" in the system which already has been measured. That is, how much information about the preparation can be extracted from the system by a second observer who does not communicate with the first observer. A further question we would like to understand is whether from the axioms of quantum theory we can obtain a "classical-like" picture when a physical system in an unknown state can be repeatedly measured, yet still retain information about the original state preparation. In what follows we analyze a simple example which illuminates these two questions.

First we specify the task of a measurement. In measurement we wish to determine some parameters of the state of a quantum system which correspond to a symmetry group. As an example, consider the position measurement which is connected with the group of translations, or measurement of the angle of orientation connected with the group of rotations. In what follows we analyze the simplest example of a continuous parameter  $\varphi \in \langle 0, 2\pi \rangle$ , the phase, which parameterizes the group of rotations in the two-dimensional space of the U(1)group. To make our discussion more physical we consider a model of optimal quantum clocks discussed in our previous work [2]. We will analyze the situation when the observers have no a priori knowledge about the original state preparation. This means that the prior phase distribution is constant and equal to  $1/2\pi$ 

In our previous paper we studied the problem of building an optimal quantum clock from an ensemble of N ions

[3]. In particular we assumed an ion trap with N two-level ions all in the ground state  $|\Psi\rangle = |0\rangle \otimes \ldots \otimes |0\rangle$ . This state is an eigenstate of the free Hamiltonian and thus cannot record time (phase). Therefore the first step in building a clock was to bring the system to an appropriate initial (reference) state  $\hat{\Omega}$  which is not an energy eigenstate. For instance, one can apply a Ramsey pulse whose shape and duration is chosen such that it puts all the ions in the product state

$$\hat{\Omega} = \hat{\rho}^{\otimes N},\tag{3}$$

with  $\hat{\rho} = |\psi\rangle\langle\psi|$  and  $|\psi\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ . this preparation stage, the ions evolve in time according to the Hamiltonian evolution  $\Omega(t) = U(t)\Omega U^{\dagger}(t)$ ,  $\hat{U}(t) = \exp\{-it\hat{H}\}\$  (we use units such that  $\hbar = 1$ ). Therefore these ions can be viewed as a time-recording device. The task is to determine this time t (or equivalently the corresponding phase) by carrying out a measurement on the ions. Note that because of the indeterminism of quantum mechanics it is impossible, given a single set of N two-level ions, to determine the elapsed time with certainty. As we have shown earlier [4] one can find an optimal measurement (see below) with the help of which information about the phase can be most optimally "extracted" from a system of N identically prepared spin-1/2 particles. The ability of the system to retain information about the phase (time) depends very much on the choice of the initial reference state  $\hat{\Omega}$ . For instance, if this state is an eigenstate of the total Hamiltonian, the system is not able to record (keep) time information. In Ref. [2] we addressed the question of which is the most appropriate initial state  $\hat{\Omega}$  of N spin-1/2 particle which "keeps" the record of phase in the most reliable way. In other words, what are the optimal quantum clocks and what is the performance of such quantum clocks when compared with classical clocks.

In the present paper we investigate another aspect of this comparison. Namely, we discuss the "robustness" of quantum clocks with respect to repeated measurements performed on them. Classical clocks, as all classical objects, do not change their state or behavior when they are observed. As we stated above this is no more true for quantum objects. This has consequences for the functioning of our proposed quantum clocks. In particular, one can ask whether quantum clocks may be robust enough in the sense that repeated readout of the time, let us say by many independent and non-communicating observers, can provide reliable information (if any) about the time to all of them. In order to find quantitative answers to our questions let us recall briefly the details of how time is read out from our quantum clocks.

In general, a quantum-mechanical measurement is described by a POVM [1,5,6] which is a set  $\{\hat{O}_r\}_{r=1}^R$  of positive Hermitian operators such that  $\sum_r \hat{O}_r = \hat{\mathbb{1}}$ . Because such measurement is in general non-deterministic, to each outcome r of the measurement we associate an estimate  $t_r$  of the time elapsed. The difference between

the estimated time  $t_r$  and the true time t is quantified by a cost function  $f(t_r - t)$  [6]. Here we note that because of the periodicity of the clock, f has to be periodic. We also take f(t) to be an even function to ensure a non-zero average. Our task is to minimize the mean value of the cost function

$$\bar{f} = \sum_{r} \int_{0}^{2\pi} \mathbf{Tr}[\hat{O}_r \; \hat{\Omega}(t)] f(t_r - t) \frac{dt}{2\pi}.$$
 (4)

Following Ref. [2] we expand the cost function in a Fourier series:

$$f(t) = w_0 - \sum_{k=1}^{\infty} w_k \cos kt. \tag{5}$$

The essential hypothesis made by Holevo [6] is the positivity of the Fourier coefficients:  $w_k \geq 0$ , (k = 1, 2, ...) Without loss of generality (see Ref. [2]) we can assume the initial (reference) state  $\hat{\Omega} = |\psi\rangle\langle\psi|$ ,  $|\psi\rangle = \sum_m a_m |m\rangle$  to be a pure state (in what follows we make a phase convention such that  $a_m$  are real and positive). In this case for the mean cost (4) we find the bound [6]

$$\bar{f} \ge w_0 - \frac{1}{2} \sum_{k=1}^{\infty} w_k \sum_{\substack{m,m' \ |m-m'|=k}} a_m a_{m'}.$$
 (6)

In this last expression, equality is attained only if the measurement is of the form

$$\hat{O}_r = p_r |\Psi_r\rangle \langle \Psi_r| \quad ; \tag{7}$$

with  $p_r \geq 0$  and  $\sum_r \hat{O}_r = \hat{1}$ , where

$$|\Psi_r\rangle = e^{it_r \hat{H}} |\Psi_0\rangle, \qquad |\Psi_0\rangle = \frac{1}{\sqrt{N+1}} \sum_{m=0}^{N} |m\rangle, \qquad (8)$$

Holevo [6] originally considered covariant measurements in which times  $t_r$  take a continuum of values between 0 and  $2\pi$ . But as shown in Ref. [4] the completeness relation can also be satisfied by taking a discrete set of times  $t_r = \frac{2\pi r}{N+1}, r = 0, ..., N$ . The states  $|\Psi_r\rangle$  form an orthonormal basis of the Hilbert space, and the corresponding measurement is therefore a von Neumann measurement. This is important for applications, because it means that it is not necessary to use an ancilla to make the optimal measurement.

We should also note that the states  $|\Psi_{\tau}\rangle$  are the eigenstates of the Pegg-Barnett Hermitian phase operator [7]. For this reason we call them "phase states". In the basis  $|m\rangle$  of the symmetric subspace of N two-level ions (spin-1/2 particles) they can be expressed as

$$|\Psi_r\rangle = \frac{1}{\sqrt{N+1}} \sum_{m=0}^{N} e^{i\frac{2\pi}{N+1}rm} |m\rangle.$$
 (9)

The fact that the optimal measurement can be chosen as a von Neumann measurement is important for our further considerations. This is due to the fact that the state immediately after the measurement is uniquely determined by the von Neumann projection postulate.

Before we proceed further we turn our attention to the fact that from the positivity of  $w_k$  it follows that not all cost functions are covered by the above result. Specifically, the quadratic deviation  $t^2$  cannot be used. On the other hand for small t it can be well approximated by the cost function  $4\sin^2\frac{t}{2}\simeq t^2$ . Therefore, in what follows we will use the cost function,  $4\sin^2t/2$ . In this case  $\bar{f}\simeq\Delta t^2$ .

Once we have determined the optimal measurement we have to specify the initial (reference) state  $\hat{\Omega}$  of our system. As discussed in Ref. [2], by an appropriate choice of this state one can substantially improve the quality of estimation. However, this concerns the estimation performed by the first observer (see footnote 1). The subsequent observers will actually always observe only rotated phase states. These are generated in the von Neumann measurement performed by the previous observer and subsequent time evolution. Therefore, in order to simplify our calculations we will assume that the initial (reference) state  $\hat{\Omega}$  is the phase state  $\hat{\Omega} = |\Psi_0\rangle\langle\Psi_0|$  given by Eq. (8).

Let us study now how the independent observers measure a system of N spin-1/2 particles initially prepared in an unknown state obtained by the rotation of the reference state (8). As far as the first is observer concerned, the problem has been already solved (see [4]) and the mean cost Eq. (4) can be calculated. For our reference state  $\hat{\Omega} = |\Psi_0\rangle\langle\Psi_0|$  the mean cost as a function of number N of spin-1/2 particles is given by the expression

$$\Delta t^2 \simeq \bar{f}(N;1) = 2 \left[ 1 - \frac{N}{N+1} \right] = \frac{2}{N+1}.$$
 (10)

We see that the mean cost when a single measurement is performed (N=1) takes the value  $\bar{f}(N=1;1)=1$ . On the contrary, for  $N\to\infty$  the mean cost is equal to zero. Specifically, for large N the variance  $\Delta t$  goes to zero as  $1/\sqrt{N}$ . This is far of being optimal<sup>1</sup>. Nevertheless, as our task is to study how much information subsequent observers can gain we are not over-worried about the optimality of the of the preparation of the reference state. Our further result can be understood as a lower bound

<sup>1</sup>As shown in Ref. [2] in order to make this variance minimal we should take the reference state to be

$$|\Psi_{opt}\rangle \simeq \frac{\sqrt{2}}{\sqrt{N+1}} \sum_{m=0}^{N} \sin \frac{\pi(m+1/2)}{N+1} |m\rangle.$$
 (11)

In this case the cost decreases for large N as  $\bar{f}_{opt} \simeq \frac{\pi^2}{(N+1)^2}$  corresponding to  $\Delta t_{opt} \simeq \frac{\pi}{(N+1)}$ .

and the optimization can be performed rather straightforwardly anyway.

Now we turn our attention to subsequent observers. We have assumed our observers do not communicate. If they do then the first observer can broadcast the result of his measurement (or, which is equivalent he can broadcast the orientation of his apparatus) and there is no need for subsequent observers to perform any measurement, because they know that they cannot perform better than this first observer. To describe the mean cost of the estimation of subsequent observers, we have to modify in Eq. (4) the conditional probability distribution  $p_1(r|t) = \mathbf{Tr}[\hat{O}_r \ \hat{\Omega}(t)]$  characterizing the measurement statistics of the observer. This is because the (k+1)-st observer does not observe the original state  $\hat{\Omega}(t)$ . He can only measure the state generated via the measurement performed by the previous k-th observer. In addition, the following random factors enter the game: Firstly, the (k+1)-st observer does not have full information about the choice of the measuring apparatus of the k-th observer. Although all observers posses the optimal measuring apparatus of the same construction (corresponding to the optimal von Neumann measurement) there is one parameter they can choose at random. Namely, if we take the POVM characterized by the set of projectors  $O_r = |\Psi_r\rangle\langle\Psi_r|, r = 0, \dots, N$  and we rotate them all by the same transformation  $\hat{U}(\alpha) = \exp\{-i\alpha \hat{H}\}$  we get a new POVM  $\hat{O}_r^{\alpha} = \hat{U}(\alpha)\hat{O}_r\hat{U}^{\dagger}(\alpha)$  which also corresponds to the optimal measuring apparatus. It is this information about the angle  $\alpha' \in (0, 2\pi)$  characterizing the "actual orientation" of the k-th measuring apparatus which is not available to the (k + 1)-st observer. The second piece of information which is not available to the (k+1)-st observer is, which of the possible outcomes r'of the measurement was detected by the k-th observer. Finally, the actual time t' when this measurement was performed is also unknown (however, as we will see in a moment, this is not important for our consideration). Taking into account these random factors the required conditional probability distribution  $p_{k+1}(r|t,\alpha)$  (we have included the parameter  $\alpha$  into the conditional probability distribution) reads

$$p_{k+1}(r|t,\alpha) = \sum_{r'=0}^{N} \int_{0}^{2\pi} p_{k}(r'|t',\alpha') \frac{d\alpha'}{2\pi}$$

$$\times \operatorname{Tr} \left[ \hat{O}_r^{\alpha} \, \hat{U}(t - t') \hat{O}_{r'}^{\alpha'} \hat{U}^{\dagger}(t - t') \right]. \tag{12}$$

It is easily seen that this can be simplified as

$$p_{k+1}(r|t,\alpha) = \mathbf{Tr} \left[ \hat{\Omega}_{k+1}(t) \ \hat{O}_r^{\alpha} \right], \tag{13}$$

where

$$\hat{\Omega}_{k+1}(t) = \sum_{r'=0}^{N} \int_{0}^{2\pi} p_k(r'|t,\alpha') \ \hat{O}_{r'}^{\alpha'} \ \frac{d\alpha'}{2\pi}.$$
 (14)

The last transformation is possible because  $p_k(r'|t', \alpha') = p_k(r'|t, \alpha' + t - t')$  and the integration with respect to  $\alpha'$  ensures that the shift (t - t') is irrelevant.

Using the iterative definition given by Eqs. (13) and (14) together with the definition for the mean cost (4) we calculate the precision of the measurement of time performed with the quantum clocks as a function of the number of qubits N and the number of subsequent observers k:

$$\Delta t^2 \simeq \bar{f}(N;k) = 2 \left[ 1 - \left( \frac{N}{N+1} \right)^k \right]. \tag{15}$$

This is the main result of our paper. We stress that the above result holds for the reference state corresponding to the phase state (8) and the case that observers have no a priori knowledge about the initial state preparation. It can be generalized to the case when the initial reference state is take to be the optimal state (11) - unfortunately in this case we are not able to find a solution in an elegant closed analytical form.

Let us summarize our result: We have shown that quantum information can be recycled in a sense, that by performing a measurement on quantum systems which have already been measured independent observers can still obtain non-trivial information about the original preparation of the quantum system (i.e. the quantum information). The larger is the ensemble (N) the more robust is the quantum system with respect to subsequent measurements. Obviously, as follows from Eq. (15) for the (k+1)-th observer the mean cost of the estimation will be larger than for the k-th observer. From the point of view of information stored in the system, in the large-N limit the quantum system behaves very classically, i.e. an infinite number of independent observers who have no prior knowledge about the state preparation can precisely measure the state of the system.

One of us (V.B.) thanks Rado Derka and Serge Massar for helpful discussions. This work was in part supported by the Royal Society, by the IST project EQUIP under the contract IST-1999-11053, by the UK Engineering and Physical Sciences Research Council, and by the CREST, Research Team for Interacting Career Electronics.

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